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Laplace Transform of Derivatives.

Theorem ① If $L\{F(t)\} = f(s)$, then

$$L\{F'(t)\} = s f(s) - F(0),$$ provided

$$\lim_{t \to \infty} \left[e^{-st} F(t) \right] = 0.$$

Proof: Let $L\{F(t)\} = f(s)$, then

$$L\{F'(t)\} = \int_0^\infty e^{-st} \frac{dF}{dt} dt$$

$$= \left[e^{-st} F(t) \right]_{t=0}^{\infty} - \int_0^\infty e^{-st} (-s) F(t) dt$$

$$= -F(0) + s \int_0^\infty e^{-st} F(t) dt$$

$$= -F(0) + s f(s)$$

$\therefore$ $L\{F'(t)\} = s f(s) - F(0)$ proved

In Generalisation

$$L\{F^n(t)\} = s^n f(s) - s^{n-1} F(0) - s^{n-2} F'(0) - \cdots - s F^{n-2}(0) - F^{n-1}(0)$$

In Particular, when $n = 2, 3, 4 \ldots$ we have

$$L\{F''(t)\} = s^2 f(s) - s F(0) - F'(0)$$

$$L\{F'''(t)\} = s^3 f(s) - s^2 F(0) - s F'(0) - F''(0)$$

$$L\{F^{iv}(t)\} = s^4 f(s) - s^3 F(0) - s^2 F'(0) - s F''(0) - F'''(0)$$

Theorem 2 Laplace Transform of Integral ②

If $L\{F(t)\} = f(s)$

then $\frac{1}{s} f(s) = L\left\{\int_0^t F(u)\,du\right\}$

proof

Let $L\{F(t)\} = G(s)$ and

$$G(t) = \int_0^t F(u)\,du \quad \text{—— (1)}$$

we have to prove that

$$L\left\{\int_0^t F(u)\,du\right\} = \frac{1}{s} f(s)$$

From (1)

$$G(0) = \int_0^0 F(u)\,du = 0$$

& $G'(t) = \frac{d}{dt}\left\{\int_0^t F(u)\,du\right\} = F(t)$

$\therefore L\{G'(t)\} = s\,L\{G(t)\} - G(0)$

or $L\{F(t)\} = s\,L\{G(t)\} - 0$

$= s\,L\{G(t)\}$

$\Rightarrow \frac{1}{s} f(s) = L\{G(t)\} = L\left\{\int_0^t F(u)\,du\right\}$

Hence proved.

Multiplication by powers of t. ③

Theorem 3 If $L\{F(t)\} = f(s)$, then

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s) \text{ for}$$

$n = 1, 2, 3 \dots$

proof Let $L\{F(t)\} = f(s)$ and

$$\frac{d^n f(s)}{ds^n} = f^n(s)$$

Then $f(s) = \int_0^\infty e^{-st} F(t)\,dt$

$$\frac{d f(s)}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} F(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}\{e^{-st} F(t)\}\,dt$$

$$= \int_0^\infty -t e^{-st} F(t)\,dt$$

or $(-1)\frac{d f(s)}{ds} = \int_0^\infty e^{-st}\{t F(t)\}\,dt$

$$= L\{t F(t)\}$$

This proves that the theorem is true for $n=1$.

Let us suppose that the theorem is true for $n = r$ so that

$$(-1)^r f^{(r)}(s) = L\{t^r F(t)\}$$

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or $(-1)^r f^{(r)}(s) = L\{t^r F(t)\}$

$$= \int_0^\infty e^{-st} t^r F(t)\,dt$$

$$(-1)^r \frac{d^{r+1}}{ds^{r+1}} f(s) = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} t^r F(t)\right) dt$$

$$= \int_0^\infty -t\, e^{-st} t^r F(t)\,dt$$

$$\Rightarrow (-1)^{r+1} f^{r+1}(s) = \int_0^\infty e^{-st} \{t^{r+1} F(t)\}\,dt$$

$$= L\{t^{r+1} F(t)\}$$

This proves that the theorem is true for $n = r+1$, if it is true for $n = r$. But we have already shown that theorem is true for $n = 1$ and hence for $n = 1+1 = 2$, & therefore $n = 3$ & so on. Hence it is true for any positive integer n.

Division by t

Theorem 9. If $L\{F(t)\} = \{f(s)\}$, then

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)\,dx$$

provided the integral exists.

proof. Let $L\{F(t)\} = f(s)$, Then

$$f(s) = \int_0^\infty e^{-st} F(t)\,dt \quad — (1)$$

Integrating this w.r.t s from $s=s$ to $s=\infty$

$$\int_s^\infty f(s)\,ds = \int_s^\infty ds \int_0^\infty e^{-st} F(t)\,dt$$

$\because$ s and t are independent variables and hence order of integration in the repeated integral can be interchanged.

$$\therefore \int_s^\infty f(s)\,ds = \int_0^\infty F(t)\,dt \int_s^\infty e^{-st}\,ds$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st}\,ds\right] F(t)\,dt$$

$$= \int_0^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty F(t)\,dt$$

$$= \int_0^\infty e^{-st} \left\{\frac{F(t)}{t}\right\} dt$$

$$= L\left\{\frac{F(t)}{t}\right\}$$ Proved.